# MATH 4030 Differential Geometry
# Tutorial 4, 4 October 2017

Verify that the following subsets of $\mathbb{R}^3$ are surfaces.

1. $S = \{(x, y, z) \in \mathbb{R}^3 | \ x^2 + y^2 + z^2 = 1\}$ (sphere)

Method 1. (covering $S$ by 6 graphs)

Let $U_{yz} = \{(y, z) \in \mathbb{R}^2 | \ y^2 + z^2 < 1\}$, and define $U_{xz}$ and $U_{xy}$ similarly. Let $V_{yz}^{\pm} = S \cap \{\pm x > 0\}$, and define $V_{xz}^{\pm}$ and $V_{xy}^{\pm}$ similarly. Then

- The six open subsets (of $S$) $V_{yz}^-, V_{yz}^+, V_{xz}^-, V_{xz}^+, V_{xy}^-, V_{xy}^+$ cover the whole $S$.
- Each of these sets is the graph of some smooth function defined on one of the $U$'s. For example, $V_{yz}^+$ is the graph of the function $f_{yz}^+ : U_{yz} \to \mathbb{R} : (y, z) \mapsto \sqrt{1 - y^2 - z^2}$.

Method 2. (using implicit function theorem)

Let $F : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$. Then $F$ is smooth and $S = F^{-1}(0)$. It suffices to show that 0 is a regular value of $F$. Let $p = (x_0, y_0, z_0)$ be a point in $F^{-1}(0)$. Then $dF(p)$ is given by the matrix $[2x_0 \ 2y_0 \ 2z_0]$. If $dF(p) = \mathbf{0}$, then $x_0 = y_0 = z_0 = 0$, and hence $F(p) = -1$, a contradiction. It follows that every point in $F^{-1}(0)$ is not a critical point. In other words, 0 is a regular value of $F$.

Method 3. (covering $S$ by 2 charts- stereographical projections)

We cover $S$ by the two open subsets $V_1 = S - \{(0, 0, 1)\}$ and $V_2 = S - \{(0, 0, -1)\}$, and show that each of them has a parametrization, which is done as follows: for $V_1$, consider the map

$$X : \mathbb{R}^2 \to V_1 : (u, v) \mapsto \left( \frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

Geometrically, $X$ maps every point $p$ in the $xy$-plane to the unique point $X(p)$ in $V_1$ such that $p, X(p)$ and the north pole $(0, 0, 1)$ are collinear. $X$ is a homeomorphism with inverse

$$X^{-1} : V_1 \to \mathbb{R}^2 : (x, y, z) \mapsto \left( \frac{x}{1 - z}, \frac{y}{1 - z} \right).$$

It suffices to check that $dX$ is one-to-one at every point in $\mathbb{R}^2$. We have

$$dX(u, v) = \begin{bmatrix} \dfrac{2(-u^2 + v^2 + 1)}{(u^2 + v^2 + 1)^2} & \dfrac{-4uv}{(u^2 + v^2 + 1)^2} \\ \dfrac{-4uv}{(u^2 + v^2 + 1)^2} & \dfrac{2(u^2 - v^2 + 1)}{(u^2 + v^2 + 1)^2} \\ \dfrac{4u}{(u^2 + v^2 + 1)^2} & \dfrac{4v}{(u^2 + v^2 + 1)^2} \end{bmatrix},$$

which has full rank because the $2 \times 2$ minors of this matrix (which determine $X_u \times X_v$) are $\dfrac{-8u}{(u^2 + v^2 + 1)^3}, \dfrac{-8v}{(u^2 + v^2 + 1)^3}$ and $\dfrac{4[1 - (u^2 + v^2)^2]}{(u^2 + v^2 + 1)^4}$ respectively, and they cannot be zero simultaneously.

For $V_2$, we consider the map

$$Y : \mathbb{R}^2 \to V_2 : (u,v) \mapsto \left( \frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{-u^2-v^2+1}{u^2+v^2+1} \right)$$

and proceed in the similar way as above.

Method 4. (spherical coordinates)

Let $X : (0,\pi) \times (0, 2\pi) \to S : (\theta, \phi) \mapsto (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Then $X$ is smooth, and its differential

$$dX(\theta,\phi) = \begin{bmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{bmatrix}$$

and

$$|X_\theta \times X_\phi| = \sin\theta \neq 0$$

(I am sure you will have to compute this later by yourself, so I skip the details). Since $|X_\theta \times X_\phi|$ is always non-zero, it follows that $dX(\theta,\phi)$ is one-to-one at every point $(\theta,\phi) \in (0,\pi)\times(0,2\pi)$. The image of $X$ is $S - \{y=0,\ x \geqslant 0\}$ which is an open subset of $S$. Furthermore, $X$ is a homeomorphism (in fact, its inverse can be expressed in terms of the Log function (with branch $0 < Arg < 2\pi$) and the inverse of the cosine function defined on $(0,\pi)$). It follows that $X$ is a parametrization of $S - \{y=0,\ x \geqslant 0\}$. Since $S = S - \{y=0,\ x \geqslant 0\} \cup S - \{z=0,\ x \leqslant 0\}$ and the latter open subset has a parametrization of the similar form, we conclude that $S$ is covered by two charts, and hence it is a surface.

2. $T = \{((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)|\ \theta,\phi \in \mathbb{R}\},\ R > r > 0$ (torus)

Method 1. (covering $T$ by a few charts defined using a periodic function in two variables)

Let $X : \mathbb{R}^2 \to T : (\theta,\phi) \mapsto ((R + r\cos\theta)\cos\phi, (R + r\cos\theta)\sin\phi, r\sin\theta)$. Then $X$ is smooth and has the property that $X(\theta + 2m\pi, \phi + 2n\pi) = X(\theta,\phi)$ for all $m,n \in \mathbb{Z}$. Its differential

$$dX(\theta,\phi) = \begin{bmatrix} -r\sin\theta\cos\phi & -(R + r\cos\theta)\sin\phi \\ -r\sin\theta\sin\phi & (R + r\cos\theta)\cos\phi \\ r\cos\theta & 0 \end{bmatrix}$$

and

$$|X_\theta \times X_\phi| = r(R + r\cos\theta) \neq 0.$$

Since $|X_\theta \times X_\phi|$ is always non-zero, it follows that $dX(\theta,\phi)$ is one-to-one at every point $(\theta,\phi) \in \mathbb{R}^2$. It can also be checked that $X$ is an open map, i.e. it maps open subsets of $\mathbb{R}^2$ to open subsets of $T$. While $X$ fails to be a homeomorphism onto its image, its restriction to some open sets $U \subseteq \mathbb{R}^2$ is, whenever $U$ is contained in a translate of the open square $(0,2\pi)\times(0,2\pi)$ (exercise). In other words, the restriction of $X$ to any open set of this type is a chart for $T$. The result thus follows by covering the square $[0,2\pi)\times[0,2\pi)$ (whose image under $X$ is the whole $T$, but which is not open) using open sets of this type. (For example, take the covering to be $\{(0,2\pi)\times(0,2\pi), (-\varepsilon,\varepsilon)\times(0,2\pi), (0,2\pi)\times(-\varepsilon,\varepsilon), (-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\}$ for small $\varepsilon > 0$.)

Method 2. (using implicit function theorem)

Let $F : \mathbb{R}^3 - \{(0,0,z)|\ z \in \mathbb{R}\} \to \mathbb{R} : (x,y,z) \mapsto (\sqrt{x^2+y^2} - R)^2 + z^2 - r^2$. Then $F$ is smooth and $T = F^{-1}(0)$. It suffices to show that 0 is a regular value of $F$. Let $p = (x_0, y_0, z_0)$ be a point in $F^{-1}(0)$. Then $dF(p)$ is given by the matrix

$$\left[2x_0 - \frac{2Rx_0}{\sqrt{x_0^2+y_0^2}},\ 2y_0 - \frac{2Ry_0}{\sqrt{x_0^2+y_0^2}},\ 2z_0\right].$$

If $dF(p) = \mathbf{0}$, then $1 - \dfrac{R}{\sqrt{x_0^2+y_0^2}} = z_0 = 0$, and hence $F(p) = -r^2$, a contradiction.

It follows that every point in $F^{-1}(0)$ is not a critical point. In other words, 0 is a regular value of $F$.